

# THE BETTI NUMBERS OF STANLEY-REISNER IDEALS OF SIMPLICIAL TREES

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## Abstract

We provide a simple method to compute the Betti numbers of the Stanley-Reisner ideal of a simplicial tree and its Alexander dual.

Keywords: resolution, monomial ideal, simplicial tree, Stanley-Reisner ideal

Simplicial trees [F1] are a class of flag complexes initially studied for the properties of their facet ideals. In this short note we give a short and straightforward method to compute the Betti numbers of their Stanley-Reisner ideals.

The *Betti numbers* of a homogeneous ideal  $I$  in a polynomial ring  $R$  over a field are the ranks of the free modules appearing in a minimal free resolution

$$0 \rightarrow \oplus_d R(-d)^{\beta_{p,d}} \rightarrow \cdots \rightarrow \oplus_d R(-d)^{\beta_{0,d}} \rightarrow I \rightarrow 0$$

of  $I$ . Here  $R(-d)$  denotes the graded free module obtained by shifting the degrees of elements in  $R$  by  $d$ . The numbers  $\beta_{i,d}$ , which we shall refer to as the  *$i$ th  $\mathbb{N}$ -graded Betti numbers* of degree  $d$  of  $I$ , are independent of the choice of the graded minimal finite free resolution.

**Definition 1** (simplicial complex). A *simplicial complex*  $\Delta$  over a set of vertices  $V = \{v_1, \dots, v_n\}$  is a collection of subsets of  $V$ , with the property that  $\{v_i\} \in \Delta$  for all  $i$ , and if  $F \in \Delta$  then all subsets of  $F$  are also in  $\Delta$ . An element of  $\Delta$  is called a *face* of  $\Delta$ . The maximal faces of  $\Delta$  under inclusion are called *facets* of  $\Delta$ . A *subcollection* of  $\Delta$  is a simplicial complex whose facets are also facets of  $\Delta$ ; in other words a simplicial complex generated by a subset of the set of facets of  $\Delta$ .  $A \subseteq V$ , the *induced subcomplex of  $\Delta$  on  $A$* , denoted by  $\Delta_A$ , is defined as  $\Delta_A = \{F \in \Delta \mid F \subseteq A\}$ .

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**Definition 2.** Let  $\Delta$  be a simplicial complex with vertex set  $x_1, \dots, x_n$  and  $R = k[x_1, \dots, x_n]$  be a polynomial ring over a field  $k$ . The *Stanley-Reisner ideal* of  $\Delta$  is defined as  $I_\Delta = (\prod_{x_i \in F} x_i \mid F \notin \Delta)$ .

**Definition 3** ([F1] leaf, joint). A facet  $F$  of a simplicial complex is called a *leaf* if either  $F$  is the only facet of  $\Delta$  or for some facet  $G \neq F$  of  $\Delta$  we have  $F \cap H \subseteq G$  for all other facets  $H$  of  $\Delta$ . Such a facet  $G$  is called a *joint* of  $F$ .

**Definition 4** ([F1] tree, forest). A connected simplicial complex  $\Delta$  is a *tree* if every nonempty subcollection of  $\Delta$  has a leaf. If  $\Delta$  is not necessarily connected, but every subcollection has a leaf, then  $\Delta$  is called a *forest*.

**Theorem 5** ([F2] Theorem 2.5). *An induced subcomplex of a simplicial tree is a simplicial forest.*

**Definition 6** (link). Let  $\Delta$  be a simplicial complex over a vertex set  $V$  and let  $F$  be a face of  $\Delta$ . The *link* of  $F$  is defined as  $\text{lk}_\Delta(F) = \{G \in \Delta \mid F \cap G = \emptyset \text{ \& } F \cup G \in \Delta\}$ .

**Lemma 7** (A link in a tree is a forest). *If  $\Delta$  is a tree and  $F$  is a face of  $\Delta$ , then  $\text{lk}_\Delta(F)$  is a forest.*

*Proof.* Suppose  $\text{lk}_\Delta(F) = \langle G_1, \dots, G_s \rangle$  where  $G_i$  is a subset of a facet  $F_i = F \cup G$  of  $\Delta$ . Now suppose  $\Gamma = \langle G_{a_1}, \dots, G_{a_r} \rangle$  is a subcollection of  $\text{lk}_\Delta(F)$ . We need to show that  $\Gamma$  has a leaf. Let  $\langle F_{a_1}, \dots, F_{a_r} \rangle$  be the corresponding subcollection of  $\Delta$ , which must have a leaf, say  $F_{a_1}$  and a joint, say  $F_{a_2}$ . Then we have  $F_{a_i} \cap F_{a_1} \subseteq F_{a_2}$  for  $i = 3, \dots, r$ . But since  $F_{a_i} = F \cup G_{a_i}$  and  $F \cap G_{a_i} = \emptyset$  for all  $i$ , we must have  $G_{a_i} \cap G_{a_1} \subseteq G_{a_2}$  for  $i = 3, \dots, r$  which means that  $G_{a_1}$  is a leaf of  $\Gamma$ .  $\square$

We will combine the above two facts with Hochster's formula for Betti numbers of the ideal and its dual [BCP].

**Theorem 8** ([BCP]). *Let  $k$  be a field and  $\Delta$  a simplicial complex over vertex set  $V$ . Then*

$$\beta_{i,j}(I_\Delta) = \sum_{A \subseteq V, |A|=j} \dim_k \widetilde{H}_{j-i-2}(\Delta_A; k) \quad (1)$$

$$\beta_{i,j}(I_\Delta^\vee) = \sum_{A \subseteq V, |A|=j} \dim_k \widetilde{H}_{i-1}(\text{lk}_\Delta(V \setminus A; k)). \quad (2)$$

If  $\Delta$  is a tree, the following theorem shows how to find Betti numbers of  $I_\Delta$ , and along the way also gives a proof of the fact that  $I_\Delta$  has a linear resolution. This last statement is not unknown, it follows also from Fröberg's characterizations of edge ideals with linear resolutions [Fr] along with observations in [HHZ], and is also proved in [CF].

**Theorem 9.** *Let  $\Delta$  be a simplicial tree with vertex set  $V$ . Then  $\Delta$  is a flag complex,  $I_\Delta$  has a linear resolution, and the Betti numbers of  $I_\Delta$  can be computed by*

$$\beta_{i,j}(I_\Delta) = \begin{cases} \sum_{A \subseteq V, |A|=j} (\text{number of connected components of } \Delta_A - 1) & j = i + 2 \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* By (1) we know that we are looking at the reduced homology modules of  $\Delta_A$  for various  $A \subseteq V$ . For a given  $A$ , we know that  $\Delta_A$  is a forest, and every connected component is a tree and therefore acyclic ([F2] Theorem 2.9). Therefore, for each such  $A$  the only possible nonzero reduced homology is the 0th one, that is when  $|A| - i - 2 = 0$  or  $|A| = i + 2$ . The formula now just follows.

In particular,  $\beta_0$  is only positive in degree 2, which implies that  $\Delta$  is a flag complex, and the fact that the resolution is linear is evident from the way the Betti numbers grow.  $\square$

**Theorem 10.** *Let  $\Delta$  be a simplicial tree with vertex set  $V$  of cardinality  $n$ . Then the  $I_\Delta^\vee$  has projective dimension 1, and its Betti numbers are*

$$\beta_{i,j}(I_\Delta^\vee) = \begin{cases} \text{number of facets of } \Delta \text{ of cardinality } n - j & i = 0 \\ \sum_{A \subseteq V, |A|=j} (\text{number of connected components of } \text{lk}_\Delta(V \setminus A) - 1) & i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* This follows from (2). Note that in this case we are looking at the homology modules of  $\text{lk}_\Delta(V \setminus A)$  for  $A \subseteq V$ . By Lemma 7  $\text{lk}_\Delta(V \setminus A)$  is a forest, and so since all the connected components are acyclic, we only have possible homology in degrees -1 (if the link is empty) and 0.

The case  $i = 1$  is the 0th homology, and we are counting the numbers of connected components minus 1, which is straightforward.

In the case  $i = 0$ , we are counting only those  $A \subset V$  where  $\text{lk}_\Delta(V \setminus A) = \{\emptyset\}$ , or equivalently  $V \setminus A$  is a facet of  $\Delta$ . So the formula for the case  $i = 0$  follows.  $\square$

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